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A PROBLEM OF THE SEQUENTIAL APPROACH TO A GROUP OF MOVING POINTS BY A THIRD-ORDER NON-LINEAR CONTROL SYSTEM†

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The problem of the fastest sequential circumvention of a group of moving points by a third-order non-linear controlled object is investigated. The necessary conditions for optimality of the control and the convergence times are used in the form of Pontryagin's Maximum Principle and the conditions for smoothing the Hamiltonian. The problem is not assumed to be divisible into several successively solvable "two-point problems".

The difference between the problem investigated here and problems previously considered is that there are not one but several target points, which must be circumvented sequentially (in time) in a given order of traversal. The complexity of the problem is that it cannot be divided into several successively solvable two-point problems, each consisting of motion from one point to the next. Here, when going from one point to another, one has to utilize information concerning all the points to be traversed in the future, since disregard of such information may seriously detract from the quality of the control [1]. Use will be made of the necessary conditions for optimality of the control and the time parameters, in the form of the Maximum Principle and conditions for smoothing the Hamiltonian, obtained for the problem of sequential control [1], of which the problem considered here is a special case. It is assumed that the controlled object goes round the points in order of increasing indices of the points (where the initial point is assigned the index zero), and that each two consecutively numbered points are fairly far apart at any time: the distance between them exceeds four minimum return radii. Under these conditions, it follows from the Maximum Principle that an optimal trajectory can consist only of arcs of circles of minimum radius and straight-line sections connecting them. At the common points of the arcs and sections, the circles and straight lines containing them are tangent to one another; the last target point is an end-point of the last section, but all other target points belong to appropriate arcs. The smoothing conditions are converted into relations from which it follows that the target points on the arcs must be in certain positions. Thus, the Maximum Principle and smoothing conditions yield a unique determination of the optimal trajectory.

A similarly formulated problem was considered in [2], where the maximum mismatch between the target points and the positions of the controlled object in a plane was minimized at certain times, with the system functioning in a given interval. Minimization was achieved by choosing the control and these times.

1. FORMULATION OF THE PROBLEM

The motion of a non-linear controlled object in a three-dimensional phase space, over a fairly long time interval $T = [t_0, t^0]$ (where t_0 and t^0 are given numbers, $t_0 < t^0$), is described by the system of equations

$$\dot{x} = \cos\alpha, \quad \dot{y} = \sin\alpha, \quad \dot{\alpha} = u; \quad |u| \leqslant 1 \tag{1.1}$$

where x, y and α are the coordinates of the phase vector of system (1.1) and u is a control parameter satisfying the above-mentioned restriction. The state (x_0, y_0, α_0) of the object (1.1) at the initial time t_0 is assumed to be given:

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad \alpha(t_0) = \alpha_0 \tag{1.2}$$

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As is well known [3, 4], system (1.1) describes the simplest motion of an aircraft (automobile) in the horizontal plane Oxy. In that case x and y are the coordinates of the aircraft, which is identified with a point in the plane, and α is the angle between the velocity vector $\mathbf{V} = (\dot{x}, \dot{y})$ and the x axis; u is a parameter characterizing the rate of change of the angle α.

This model has been used to formulate both game-theoretical [5–11] and control problems [12–19]. In particular, an optimal control has been synthesized to steer an aircraft in minimum time from an initial position to a fixed point of the plane of motion [12]. The same problem has been solved for a more complex model of the motion, described by a non-linear fourth-order system [13].

Earlier publications [3–19] have, as a rule, considered a system of equations that differs from (1.1) in having a constant coefficient V_0 on the right of the first two equations and a constant coefficient K on the right of the third. These coefficients may be avoided, however, by compression or extension of the coordinates x, y and the time t.

In what follows we shall assume that the time t^0, $t^0 > t_0$, is fairly long.

We will choose as the class of admissible controls the set $\mathbf{U}$ of all piecewise-continuous (right-continuous) scalar functions $U : T \to \{u : |u| \leq 1\}$. Every control $U \in \mathbf{U}$ generates a motion beginning at the initial point (x_0, y_0, α_0), which we will denote by

$$(x_u, y_u, \alpha_u) = ((x_u(t), y_u(t), \alpha_u(t)), \quad t \in T)$$

We will mean by a trajectory of system (1.1), generated by a control U, the set $((x_u(t), y_u(t)), t \in T)$ in the Oxy plane.

Let us consider a given group of points in the Oxy plane, say $W_i(t)$ (in what follows, unless otherwise stated, $i = 1, 2, \ldots, m$), moving long known trajectories

$$W_i(t) = g_i(t), \quad t_0 \leq t \leq t^0$$

($g_i : T \to R^2$ are given vector-valued functions). To fix our ideas and to simplify the calculations, let us assume that each point $W_i(t)$ is moving uniformly along a corresponding straight line L_i passing through the points (x_0, y_0) and $W_i(t_0)$, increasing its distance from the point (x_0, y_0). In that case the vector-valued function $g_i(t)$ may be expressed as

$$g_i(t) = W_i(t_0) + \mathbf{v}_i(t - t_0), \quad t_0 \leq t \leq t^0$$

where $\mathbf{v}_i$ is a given two-dimensional velocity vector, directed from the point (x_0, y_0) to the point $W_i(t_0)$. Suppose v_{i1}, v_{i2}, v_i are the coordinates and magnitude of this vector, $v_i < 1$, β_i is the angle between the Ox axis and the vector $\mathbf{v}_i$, and $X_i(t)$ and $Y_i(t)$ are the coordinates of the point $W_i(t)$ at time t, $t \in T$.

We shall assume that the distance between two neighbouring points $W_i(t_0)$ and $W_{i+1}(t_0)$ ($i = 0, 1, \ldots, m - 1$) is greater than four units.

Definition. We shall say that a control $U \in \mathbf{U}$ makes system (1.1) approach the points $W_i(t)$ if times t_i exist such that

$$x_u(t_i) = X_i(t_i), \quad y_u(t_i) = Y_i(t_i)$$

The *initial problem* is to determine a control $U \in \mathbf{U}$ that makes system (1.1) approach the points $W_i(t)$ in the least time, and to determine the time t_i at which this is achieved.

We will define the order of approach by the relation

$$t_0 \leq t_1 \leq \ldots \leq t_m \leq t^0 \tag{1.3}$$

that is, the points are serviced in increasing order of their indices.

2. THE NECESSARY CONDITIONS FOR OPTIMALITY OF THE CONTROLS AND TIMES

In what follows we shall use the necessary conditions for optimality of a programmed control and the choice of the time parameters in the form of Pontryagin's Maximum Principle [20], as well as the conditions for smoothing the Hamiltonian, established in [1] for the problem of sequential control. For completeness, we present them here, except that as admissible controls we will use the set of piecewise-continuous functions, and not the set of generalized programmed control-measures [21–25]. The latter were used in [1] to guarantee the existence of an optimal control and the correct satisfaction of the boundary and intermediate conditions. In what follows, the existence of a piecewise-continuous control in the main problem will be assumed a priori.

Let us recall the previous investigation [1] of the motion of a control system in Euclidean n-space X over a given time interval T (see Section 1), described by a vector differential equation

$$\dot{x} = f(t,x,u), \quad x(t_0) = x_0; \quad u(t) \in P, \quad t \in T \tag{2.1}$$

where $x \in X$ is the phase vector of system (2.1), u is an r-dimensional control parameter satisfying the above geometrical constraint, $P \subset R^r$ is a compact set, and n and r are given natural numbers. As usual [21–25], the function $f: T \times X \times P \to X$ is subject to three conditions: joint continuity with respect to all its variables, the existence of continuous partial derivatives $\partial f_i/\partial x_j$ (where x_j are the coordinates of the vector x and f_i are the coordinates of the vector-valued function $f(j = 1, 2, \ldots, n)$), and continuability of the solutions, meaning that a number $a > 0$ exists for which

$$\| f(t,x,u)\|_n \leq a(1+\| x \|_n) \quad \forall t \in T, \quad x \in X, \quad u \in P$$

($\|x\|_n$ is the Euclidean norm of a vector $x \in R^n$).

Suppose R is the real line, $\mathbb{T}$ is the set of all m-dimensional vectors $\mathbf{t}$ whose coordinates t_i satisfy the constraint (1.3), $\mathbf{U}$ is the set of all piecewise-continuous (right continuous) r-dimensional vector-valued functions $U: T \to P$; $\varphi_u = (\varphi_u(t), t \in T)$ is the motion beginning at the initial point $x(t_0)$ generated by a control $U \in \mathbf{U}$, $S(t_i^*, t \mid U^*)$ is the fundamental solution matrix of the variational system [26]

$$\dot{y} = -L'(t)y \quad \left(L(t) = \left.\frac{\partial f(t,x,u)}{\partial x}\right|_{u=U^*(t),\, x=\varphi_{u^*}(t)} \right)$$

for the motion φ_{u*} generated by a control U^* (t_i^* being some instant of time, $t < t_i^*$); and $\Phi_i: T \times X \to R$; $K_i: T \times X \to R^s$ ($s < n$) are functions that are continuous and continuously differentiable (smooth) with respect to the set of variables. We assume

$$J(\mathbf{t}, U) = \sum_{i=1}^{m} \Phi_i(t_i, \varphi_u(t_i)) \quad (\mathbf{t}, U) \in \mathbb{T} \times \mathbf{U}$$

Let $\mathbf{W}$ denote the set of all pairs $(\mathbf{t}, U) \in \mathbb{T} \times \mathbf{U}$ satisfying the relations

$$K_i(t_i, \varphi_u(t_i)) = 0$$

It is assumed that this set is not empty.

The fundamental *problem* is to minimize the quality criterion $J(\mathbf{t}, U)$ over the set $\mathbf{W}$. In other words, it is required to select a pair $(\mathbf{t}, U)$ which will steer the controllable system onto the manifold

$$\mathbf{M}_i = ((t,x): \ K_i(t,x) = 0)$$

at times t_i in such a way that the sum of the values of the function Φ_i, evaluated at the points $(t_i, \varphi_u(t_i))$, is a minimum.

Similar problems, in a game-theoretical setting, were considered in [27–29].

Theorem 1. Let (t^*, U^*) be the optimal solution of the main problem and let $t_i^* < t_{i+1}^*$ for any $i = 1, 2, \ldots, m-1$. Then vectors $\Lambda_i^*, \Lambda_i^* \in R^s$ exist for which the functions $\overline{\psi}_k^*(t)$ defined by the relations

$$\psi_i^*(t) = (l_i^*)' S(t_i^*, t \mid U^*), \quad \overline{\psi}_k^*(t) = \sum_{i=k}^{m} \psi_i^*(t) \tag{2.2}$$

$$l_i^* = -\operatorname{grad} \Phi_i(t_i^*, \varphi_{u^*}(t_i^*)) - \left.\frac{\partial K_i(t_i^*)}{\partial x}\right|_{(t_i^*, \varphi_{u^*}(t_i^*))} \Lambda_i^*$$

for any $k = 1, 2, \ldots, m$ satisfy the following relation almost everywhere in the interval $[t_{k-1}, t_k]$

$$(\overline{\psi}_k^*(t))'f(t,\varphi_{u^*}(t),U^*(t)) = \max_{u\in P}(\overline{\psi}_k^*(t))'f(t,\varphi_{u^*}(t),u) \tag{2.3}$$

and the following equality at the times t_k^* ($k = 1, 2, \dots, m-1$)

$$\max_{u\in P}(\overline{\psi}_{k+1}^*(t_k^*))'f(t_k^*,\varphi_{u^*}(t_k^*),u) =$$
$$= \max_{u\in P}(\overline{\psi}_k^*(t_k^*))'f(t_k^*,\varphi_{u^*}(t_k^*),u) - \left.\frac{\partial\Phi_k}{\partial t}\right|_{(t_k^*,\varphi_{u^*}(t_k^*))} - \left.\frac{\partial K_k}{\partial t}\right|_{(t_k^*,\varphi_{u^*}(t_k^*))}\Lambda_k^* \tag{2.4}$$

The latter relations will henceforth be called the smoothing conditions.

If one adds the equation

$$\dot{x}_{n+1} = 1, \quad x_{n+1}(t_0) = t_0$$

to system (2.1) and then sets

$$\Phi_m = x_{n+1}(t_m), \quad \Phi_i \equiv 0, \quad i = 1,2,\dots,m-1$$

the fundamental problem becomes a time-optimal problem. Thus, Theorem 1 may be used in solving the time-optimal problem of a non-linear system circumventing a set of points $W_i(t)$.

We first note that the function $\psi_i^*(t) = (l_i^*)'S(t_i^*, t\,|\,U^*)$ may be expressed in standard form [20] as a solution of the vector equation

$$\dot{\psi}_i = -L'(t)\psi_i \tag{2.5}$$

with boundary condition $\psi_i(t_i) = l_i^*$ (2.2).

Indeed, it is well known (see [24, p. 134]) that the matrix $S(t_i^*, t\,|\,U^*)$ satisfies the equation

$$\frac{dS'(t_i^*,t\,|\,U^*)}{dt} = -L'(t)S'(t_i^*,t\,|\,U^*) \tag{2.6}$$

Multiplying the right- and left-hand sides of the last equation by l_i^* and taking the equation $S'(t_i^*, t\,|\,U^*)l_i^* = (l_i^*)'S(t^*, t\,|\,U^*)$ into consideration, we conclude that the function ψ_i^*, defined by formula (2.2), satisfies Eq. (2.5) with boundary condition $\psi_i^*(t_i^*) = l_i^*$.

3. CONSTRUCTION OF THE OPTIMAL TRAJECTORY

Pontryagin's Maximum Principle was used in the past [12, 13] to solve the time-optimal problem of steering system (1.1) from the initial position (1.2) to the origin $O = (0, 0)$ of the Oxy plane. It has been shown [12, 13] that if the origin lies outside the discs C_0^+, C_0^- bounded by circles C^+ and C^- of unit radius touching the straight line

$$(x - x_0)\sin\alpha_0 - (y - y_0)\cos\alpha_0 = 0 \tag{3.1}$$

at the point (x_0, y_0), then the optimal trajectory (OT) consists of an arc of the circle C^+ or an arc of the circle C^- and a segment of a straight line. Note that this is precisely the structure of the last sector of an OT between the points W_{m-1} and W_m. But if the origin O lies in one of the discs C_0^+, C_0^-, then the OT will consist of two arcs of circles. To be precise: if $O \in C_0^+$, the OT will be the union of an arc of C^- and an arc of a circle C_3 touching C^- and passing through the origin.

The simplicity of the solution of this problem may be explained as follows. First, for the initial system (1.1) the vector equation (2.5) is equivalent to a system of scalar equations

$$\dot{\psi}_{i1} = 0, \quad \dot{\psi}_{i2} = 0, \quad \dot{\psi}_{i3} = \psi_{i1}\dot{y} - \psi_{i2}\dot{x} \tag{3.2}$$

in the coordinates of the vector ψ_i. Second, for $i = m = 1$ the structure of the optimal control is defined by

$$u = \operatorname{sign}\psi_{13}, \quad \psi_{13} \neq 0$$

Third, system (3.2) has an analytical solution

$$\psi_{i1} = c_1, \quad \psi_{i2} = c_2, \quad \psi_{i3} = c_1 y - c_2 x + c_3$$

(c_1, c_2 and c_3 are constants of integration, yet to be determined). Moreover, it follows from the transversality condition that $c_3 = 0$. Thus, the straight line $c_1 y - c_2 x = 0$ divides the entire half-plane of motion into two parts; for motion in one of them, $u = 1$, and for motion in the other, $u = -1$.

To reduce the initial problem to the fundamental problem, we introduce an additional equation

$$\dot{z} = 1, \quad z(t_0) = t_0$$

For any $i = 1, \dots, m-1$ the functions $\Phi_i: T \times R^4 \to R$ are defined as identically zero, while the function $\Phi_m(t, x, y, \varphi, z)$ is put equal to $z - t_0$. In the case considered K_i are two-dimensional functions whose components K_{ij} ($j = 1, 2$) are defined by the relations

$$K_{i1} = x - (X_i(t_0) + (z - t_0)\nu_{i1}, \quad K_{i2} = y - (Y_i(t_0) + (z - t_0)\nu_{i2})$$

Therefore, the vectors Λ_i^* occurring in the formulation of Theorem 1, which have to be defined, will be two-dimensional. Their coordinates will be denoted by $-\Lambda_{i1}$, $-\Lambda_{i2}$, Then the vectors l_i^* (see Theorem 1) will be defined by the relations

$$l_m^* = (\Lambda_{m1}, \Lambda_{m2}, 0, -\Lambda_{m1}\nu_{m1} - \Lambda_{m2}\nu_{m2} - 1)'$$
$$l_i^* = (\Lambda_{i1}, \Lambda_{i2}, 0, -\Lambda_{i1}\nu_{i1} - \Lambda_{i2}\nu_{i2})', \quad i = 1, 2, \dots, m-1$$

where the prime denotes transposition.

As before, we let ψ_{ij} ($j = 1, 2, 3, 4$) denote the coordinates of the vector-valued function $\psi_i(t)$, $t \in [t_0, t_1]$. In the case under consideration, the variation of the function $\psi_i(t)$ over the time interval $[t_0, t_1]$ is described by a system consisting of Eqs (3.2) plus the equation $\dot{\psi}_{i4} = 0$, with boundary conditions

$$\psi_{i1}(t_i) = \Lambda_{i1}, \quad \psi_{i2}(t_i) = \Lambda_{i2}, \quad \psi_{i3}(t_i) = 0$$
$$\psi_{i4} = \begin{cases} -\Lambda_{i1}\nu_{i1} - \Lambda_{i2}\nu_{i2}, & i \neq m \\ -\Lambda_{i1}\nu_{i1} - \Lambda_{i2}\nu_{i2} - 1, & i = m \end{cases} \tag{3.3}$$

where Λ_{ij} ($j = 1, 2, \dots, 4$) are as yet undetermined constants of integration.

It is obvious that for any control $U \in \mathbf{U}$ and any time $t \in [t_0, t_1]$

$$\psi_{i3}(t) = \Lambda_{i1}\left(y_u(t) - Y_i(t_i)\right) - \Lambda_{i2}\left(x_u(t) - X_i(t_i)\right) \tag{3.4}$$

Let $\overline{\psi}_{kj}$ ($k = 1, 2, \dots, m$; $j = 1, 2, 3, 4$) be the coordinates of the vector $\overline{\psi}_k^*$ (2.2). It follows from expressions (2.2) and (3.4) that (in what follows, summation is from $i = k$ to $i = m$)

$$\overline{\psi}_{k3} = \Sigma\Lambda_{i1}\left(y_u - Y_i(t_i)\right) - \Lambda_{i2}\left(x_u - X_i(t_i)\right), \quad \overline{\psi}_{k1} = \Sigma\Lambda_{i1}, \quad \overline{\psi}_{k2} = \Sigma\Lambda_{i2} \tag{3.5}$$

In accordance with the Maximum Principle (2.3), the control active in the time interval $[t_{k-1}, t_k]$ is the control U defined by

$$U(t) = \operatorname{sign} \overline{\psi}_{k3}(t), \quad \overline{\psi}_{k3}(t) \neq 0 \tag{3.6}$$

Define

$$p_k = \Sigma\left(\Lambda_{i2} X_i(t_i) - \Lambda_{i1} Y_i(t_i)\right)$$

Then (see (3.5))

$$\overline{\psi}_{k3}(t) = \overline{\psi}_{k1} y_u(t) - \overline{\psi}_{k2} x_u(t) + p_k$$

In view of relations (3.3)–(3.6), the straight line

$$S_k = \left\{(x, y): \ \overline{\psi}_{k1} y - \overline{\psi}_{k2} x + p_k = 0\right\} \tag{3.7}$$

divides the entire plane of motion into two parts, in one of which (over the time interval $[t_{k-1}, t_k]$) the control is $U(t) = 1$, and in the other, $U(t) = -1$. Motion is also possible along the straight line S_k of (3.7), under the action of the control $U(t) \equiv 0$, which is a singular control for the problem under investigation here [22].

Using the Maximum Principle – relation (2.3) and its corollary (3.6), one can show that an OT is the union of arcs D_i ($i = 0, 1, \ldots, m-1$) of circles of unit radius and segments G_k ($k = 1, 2, \ldots, m$) of straight lines (3.7) connecting them; in these trajectories, the arcs and segments are tangent to one another at their common points. In addition, the first part of an OT is generally an arc D_0, and the last part is a segment G_m ending at the point W_m; the number of arcs is equal to the number of moving targets, and the encounter of the controlled object with the targets $W_i(t)$ occurs on the corresponding arcs D_i, that is, $W_i(t_i) \in D_i$ ($i = 1, 2, \ldots, m-1$). The proof of this proposition is based on establishing, using relation (3.6), that any other structure of an OT is impossible.

We shall now show that the position of the point $W_i(t_i)$ on the appropriate arc D_i is uniquely defined by the smoothing condition (2.4), which implies that the vectors $\mathbf{V}(t_i) - \mathbf{v}_i(t_i)$, Λ_i^*, where $\mathbf{V}(t_i)$, and $\mathbf{v}_i(t_i)$ are the velocity vectors of the controlled object and the ith target, respectively, at the point of encounter, must be orthogonal; the vector Λ_i^* was defined in Theorem 1.

We first observe that in the case under consideration the functions $\Phi_i: R^{n+1} \to R$; $K_i: R^{n+1} \to R^2$ do not depend explicitly on time. Hence the last two terms on the right-hand side of (2.4) must be omitted. In the initial and auxiliary systems (1.1) and (3.2), the smoothing conditions (2.4) have the form

$$\overline{\psi}_{k+11} \cos\alpha_u(t_k^*) + \overline{\psi}_{k+12} \sin\alpha_u(t_k^*) + \left|\overline{\psi}_{k+13}(t_k^*)\right| = \overline{\psi}_{k1} \cos\alpha_u(t_k^*) +$$

$$+ \overline{\psi}_{k2} \sin\alpha_u(t_k^*) + \left|\overline{\psi}^*_{k3}(t_k^*)\right| - (\upsilon_{k1}\Lambda_{k1} + \upsilon_{k2}\Lambda_{k2}), \quad i = 1, 2, \ldots, m-1 \tag{3.8}$$

(v_{k1} and v_{k2} are the coordinates of the velocity vector $\mathbf{v}_k$ of the kth target).

In view of relations (3.4) and (3.5), we have the following equalities

$$\psi_{k3}(t_k^*) = 0, \quad \overline{\psi}_{k3}(t_k^*) = \overline{\psi}_{k+13}(t_k^*) + \psi_{k3}(t_k^*), \quad \left|\overline{\psi}_{k+13}(t_k^*)\right| = \left|\overline{\psi}_{k3}(t_k^*)\right| \tag{3.9}$$

It follows from relations (3.8) and (3.9) that

$$\Lambda_{k1}\left(\cos\alpha_u(t_k^*) - \upsilon_{k1}\right) + \Lambda_{12}\left(\sin\alpha_u(t_k^*) - \upsilon_{k2}\right) = 0 \tag{3.10}$$

Equality (3.10) has the following geometrical meaning: at the time t_k^* the controlled object encounters the kth target, the velocity vectors $\mathbf{V}(t) - \mathbf{v}_k(t)$, Λ_k^* are orthogonal, which it was required to prove.

Remark. If the target points $W_i(t)$ are stationary, this fact can be used to show that the points $W_i(t)$ divide the corresponding arcs D_i into two equal parts.

4. EXAMPLE

We shall indicate formulae to construct optimal trajectories of system (1.1) when $m = 1$ and $m = 2$. To simplify the calculations we shall assume, without loss of generality, that $t_0 = x_0 = y_0 = \alpha_0 = 0$.

For $m = 1$, the OT will consist of an arc D_0 of a circle and a section of a straight line. The length α of the arc of this OT may be determined from the equation

$$\lambda_{mb} + \alpha = \lambda_{m1} / \upsilon_1 \tag{4.1}$$

where

$$\lambda_{mb} = \sqrt{(x_m - x_b)^2 + (y_m - y_b)^2}, \quad \lambda_{m1} = \sqrt{(x_m - x_1)^2 + (y_m - y_1)^2}$$

$$x_m = \frac{x_b \sin\alpha - y_b \cos\alpha}{\sin\alpha - \operatorname{tg}\beta_1 \cos\alpha}, \quad y_m = x_m \operatorname{tg}\beta_1; \quad \beta_1 \neq \frac{\pi}{2} \tag{4.2}$$

$$x_b = \sin\alpha, \quad y_b = 1 - \cos\alpha$$

$x_1 = X_1(0)$, $y_1 = Y(0)$ are the coordinates of the point $W_1(t)$ at time $t = 0$.

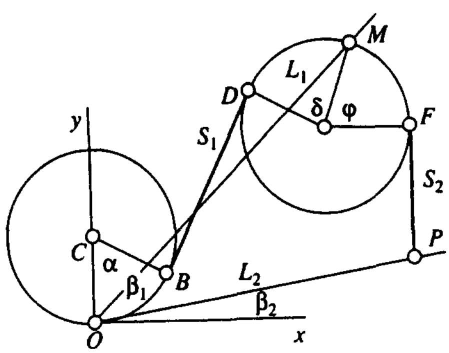

Fig. 1

Equation (4.1) was derived in the following way. One first determines the coordinates x_m, y_m of the point M at which the straight lines

$$L_1 : y\cos\beta_1 = x\sin\beta_1 \tag{4.3}$$

$$S_1 : \ (x - x_b)\sin\alpha - (y - y_b)\cos\alpha = 0 \tag{4.4}$$

intersect (the straight line S_1 is tangent at $B = (x_b, y_b)$ to the circle that contains the arc D_0). One then equates the time $t_1 = \lambda_{m1}/v_1$ of motion of the first target between the points $W_1(0)$ and M to the time $t = \lambda_{mb} + \alpha$ of motion of the controlled object between the points $W_0(W_0 = (0, 0))$ and M along the arc and the section.

Now consider the case when $m = 2$. These are four possible relative positions of the straight lines L_1 and L_2 determined by the angles β_1 and β_2 and the velocity vector of the controlled object at the starting time. We shall consider only one of them, say (see Fig. 1)

$$0 \leqslant \beta_1 \leqslant \pi, \quad \beta_1 - \pi \leqslant \beta_2 \leqslant \beta_1 \tag{4.5}$$

Auxiliary problem 1. At a given time $t, t \geqslant t_1$ ($t_1 = \lambda_{m1}/v_1$) of convergence of the controlled object to the first moving target, it is required to construct a convergence trajectory consisting of two arcs of unit circles and a straight-line section connecting them.

Solution. Let α be the angle of inclination of the straight line S_1 containing the first straight-line section of the desired trajectory. Then, first, α is the length of the first arc of the desired trajectory; second, the coordinates x_b, y_b of the point B at which the first arc touches the straight line S_1 (Fig. 1) are determined by the last two formulae of (4.2); third, the equation of the straight line S_1 has the form (4.4). The coordinates x_m, y_m of the point M at which the controlled object encounters the first target are given by the formulae

$$x_m = X_1(t_0) + v_1 t\cos\beta_1, \quad y_m = Y_1(t_0) + v_1 t\sin\beta_1$$

Then

$$d = \left|(y_m - y_b)\cos\alpha - (x_m - x_b)\sin\alpha\right| \tag{4.6}$$

is the distance from M to the straight line S_1. The length δ of the second arc of the trajectory is defined by the relations

$$\delta : \cos\delta = 1 - d, \quad \sin\delta = \sqrt{d(2-d)} \tag{4.7}$$

The coordinates x_d, y_d of the point D at which the second arc touches the straight line S_1 and the length $|\mathrm{BD}|$ of the straight-line section of the trajectory are given by the formulae

$$x_d = x_m - d\sin\alpha - \sin\delta\cos\alpha, \quad y_d = y_m + d\cos\alpha - \sin\delta\sin\alpha \tag{4.8}$$

$$|BD| = \lambda_{db}$$

Since the velocity of motion of the controlled object along its trajectory is unity, then, equating the length of the whole trajectory to the time of motion, we obtain the equation

$$\alpha + |BD| + \delta = t \tag{4.9}$$

which may be used to determine the angle α. Having evaluated this angle, one obtains the solution of Auxiliary Problem 1 in the case (4.5), since the coordinates of the characteristic points B, M and D of the trajectory are defined by formulae (4.2) and (4.8).

The parameters of the trajectory we have found depend on the value of t. Let us choose the time t of encounter with the first target in such a way that the trajectory W_0BDM is part of the trajectory which solves the following problem.

Auxiliary problem 2. In the case (4.5), it is required to determine a trajectory along which system (2.1) converges to two moving points W_1 and W_2, consisting of two arcs of circles and two straight line sections and having the least possible length.

Let φ be the length of the second part of the second arc of the desired trajectory; $\psi = \alpha - \delta - \varphi$. Then $\delta + \varphi$ is the total length of that arc. The coordinates x_f, y_f of the point F at which that arc touches the final straight-line section of the trajectory are defined by

$$x_f = x_b + |BD|\cos\alpha + \sin\alpha - \sin\psi,$$

$$y_f = y_b + |BD|\sin\alpha - \cos\alpha + \cos\psi$$

The straight line S_2 containing the second straight-line section is described by the equation

$$S_2: \ (y - y_f)\cos\psi - (x - x_f)\sin\psi = 0$$

The coordinates x_p, y_p of the point P at which this straight line intersects the straight line

$$L_2: y = x \operatorname{tg}\beta_2$$

are evaluated by the formulae

$$x_p = y_f\cos\psi - x_f\sin\psi/(\cos\psi\operatorname{tg}\beta_2 - \sin\psi), \quad y_p = x_p\operatorname{tg}\beta_2$$

To determine the time of encounter t_2 of the controlled object with the second target and the angle φ, we have two equations

$$x_2 + v_2 t_2 \cos\beta_2 = x_p, \quad \lambda_{pf} = t_2 - t - \varphi$$

Eliminating the parameter t_2, from these equations, we obtain an equation

$$\lambda_{pf} = (x_p - x_2)/(v_2\cos\beta_2) - t - \varphi \tag{4.10}$$

for determining the angle φ. Calculating φ we get

$$t_2 = (x_p - x_2)/(v_2\cos\beta_2)) \tag{4.11}$$

As shown earlier, a necessary condition for the trajectory thus constructed to be optimal is the orthogonality of the vectors $\mathbf{V}(t) - \mathbf{v}_1(t)$ and Λ_1^*, where $\mathbf{V}(t)$ and $\mathbf{v}_1(t)$ are the velocity vectors of the controlled object and of the first target, respectively, at their point of encounter M; the vector Λ_1^* was defined in Theorem 1.

It can be shown that the required condition is satisfied if the time t at which the controlled object meets the first target is determined from the equation

$$v_1 s^{-1/2}\sin(\delta + \beta_1 - \alpha) - \cos(\gamma + \delta) = 0 \tag{4.12}$$

where

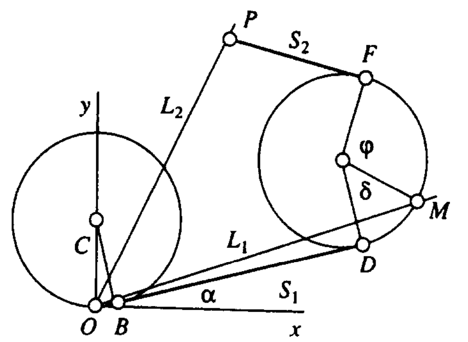

Fig. 2

$$\gamma = \operatorname{arctg} \frac{d}{\operatorname{tg}[(\varphi+\delta)/2] - \sin\delta}, \quad s = 1 + \upsilon_1^2 - 2\upsilon_1 \cos(\delta + \beta_1 - \alpha)$$

The angle α is determined as a function of the parameter t from Eq. (4.9). The distance d and the angle δ, as functions of t and α, are determined from relations (4.6) and (4.7), and the angle φ is found from Eq. (4.10). Having evaluated the time t, we solve Auxiliary Problem 2. In the case (4.5) under consideration the trajectory constructed is indeed optimal, since there is a unique time t satisfying Eq. (4.12).

Figure 2 shows the form of the optimal trajectory in the case when

$$0 \leqslant \beta_1 \leqslant \pi, \quad \beta_1 \leqslant \beta_2 \leqslant \beta_1 + \pi$$

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